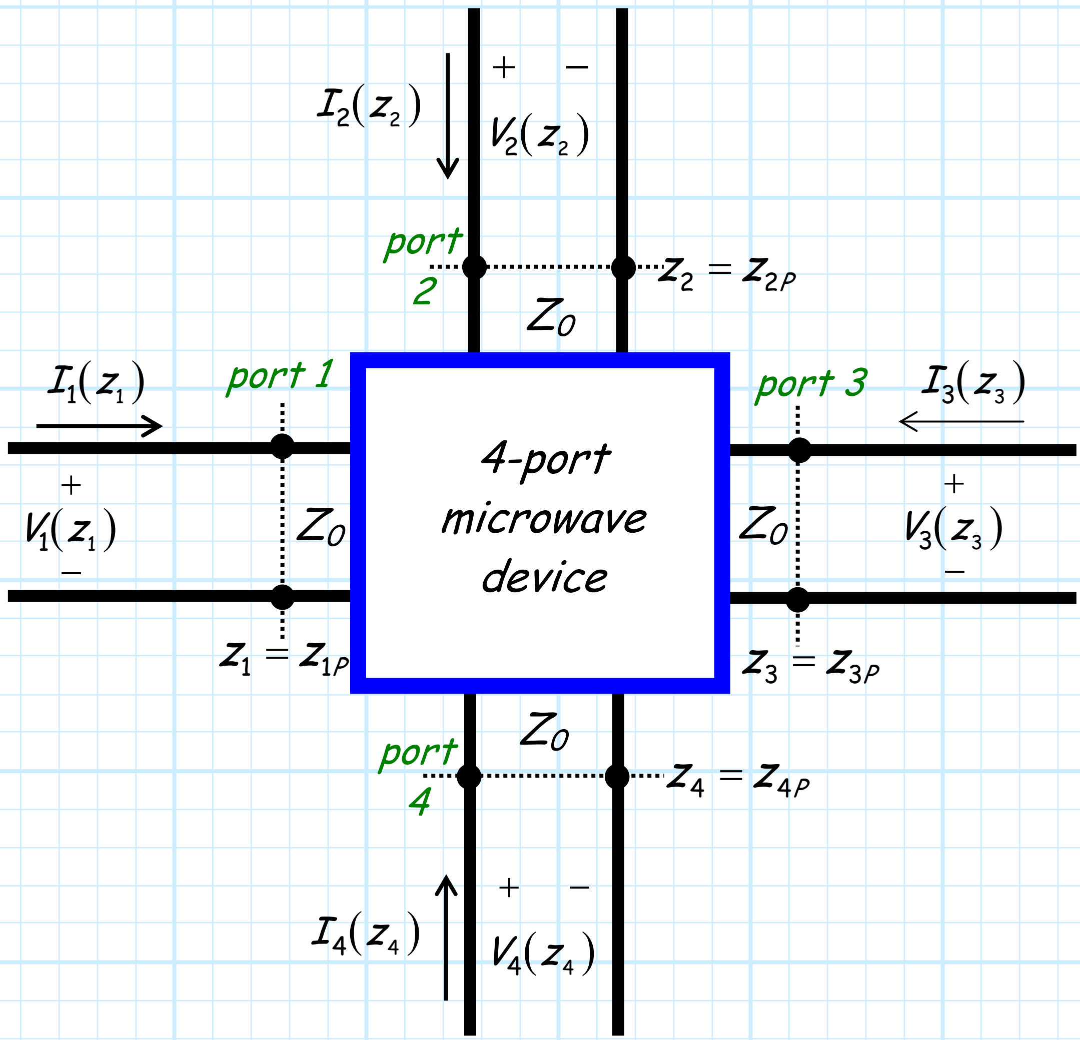
The Impedance Matrix

Consider the **4-port** microwave device shown below:



Note in this example, there are four **identical** transmission lines connected to the same "box". Inside this box there may be a very **simple** linear device/circuit, **or** it might contain a very large and **complex** linear microwave system.

→ Either way, the "box" can be fully characterized by its **impedance matrix**!

First, note that each transmission line has a specific **location** that effectively defines the **input** to the device (i.e., z_{1P}, z_{2P}, z_{3P}, z_{4P}). These often arbitrary positions are known as the **port** locations, or port **planes** of the device.

Thus, the **voltage** and **current** at port n is:

$$V_n(z_n = z_{nP}) \qquad\qquad I_n(z_n = z_{nP})$$

We can **simplify** this cumbersome notation by simply **defining** port n current and voltage as I_n and V_n:

$$V_n = V_n(z_n = z_{nP}) \qquad\qquad I_n = I_n(z_n = z_{nP})$$

For **example**, the current at port **3** would be $I_3 = I_3(z_3 = z_{3P})$.

Now, say there exists a non-zero current at **port 1** (i.e., $I_1 \neq 0$), while the current at all **other** ports are known to be **zero** (i.e., $I_2 = I_3 = I_4 = 0$).

Say we measure/determine the **current** at port **1** (i.e., determine I_1), and we then measure/determine the **voltage** at the port **2** plane (i.e., determine V_2).

The complex ratio between V_2 and I_1 is know as the **trans-impedance parameter** Z_{21}:

$$Z_{21} = \frac{V_2}{I_1}$$

Likewise, the trans-impedance parameters Z_{31} and Z_{41} are:

$$Z_{31} = \frac{V_3}{I_1} \qquad \text{and} \qquad Z_{41} = \frac{V_4}{I_1}$$

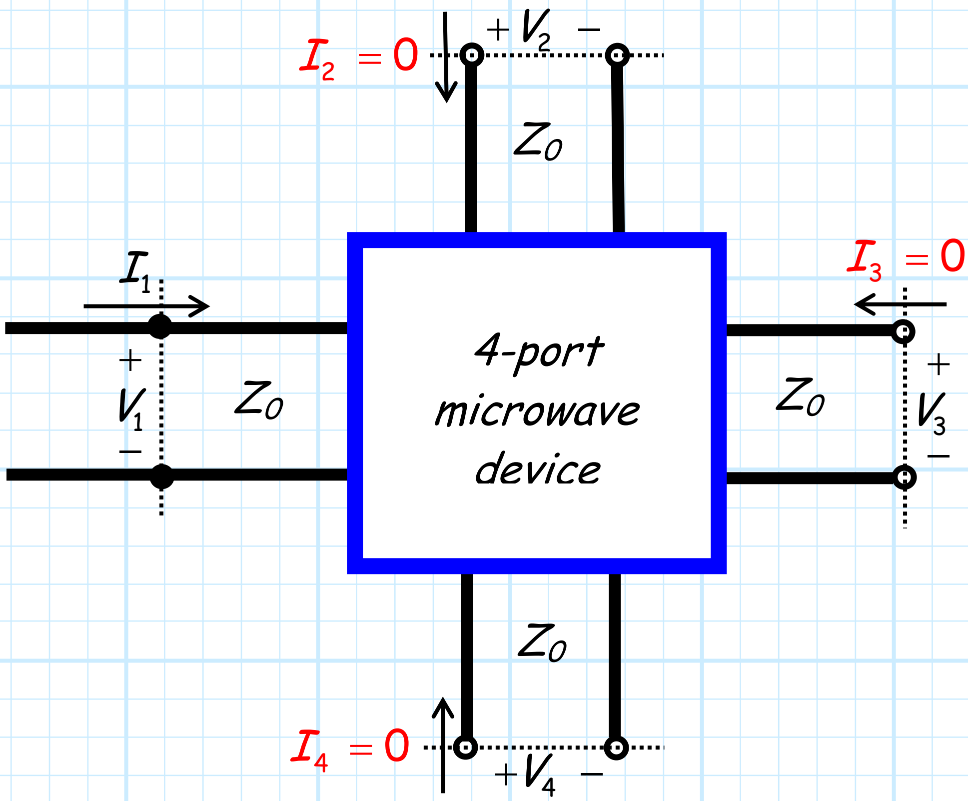
We of course could **also** define, say, trans-impedance parameter Z_{34} as the ratio between the complex values I_4 (the current into port 4) and V_3 (the voltage at port 3), given that the current at all other ports (1, 2, and 3) are zero.

Thus, more **generally**, the ratio of the current into port n and the voltage at port m is:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that } I_k = 0 \text{ for all } k \neq n\text{)}$$

Q: *But how do we ensure that all but **one** port current is zero ?*

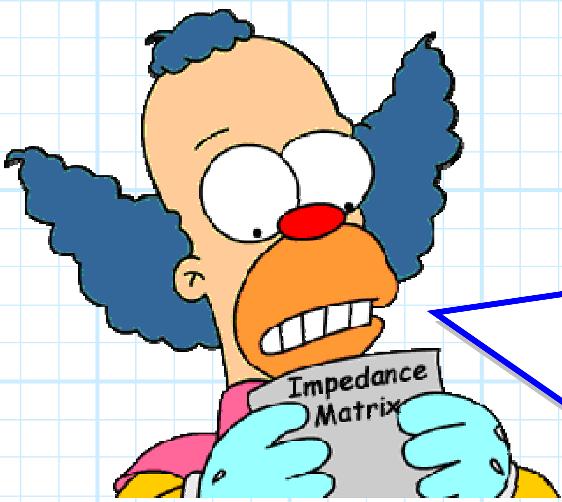
A: Place an **open** circuit **at** those ports!



Placing an **open** at a port (and it must be **at** the port!) **enforces** the condition that $I = 0$.

Now, we can thus **equivalently** state the definition of trans-impedance as:

$$Z_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{open}\text{)}$$



Q: *As impossible as it sounds, this handout is even more* ***boring*** *and* ***pointless*** *than any of your previous efforts.* ***Why*** *are we studying this? After all, what is the likelihood that a device will have an* ***open*** *circuit on* ***all*** *but one of its ports?!*

A: OK, say that **none** of our ports are **open-circuited**, such that we have currents **simultaneously** on **each** of the **four** ports of our device.

Since the device is **linear**, the voltage at any **one** port due to **all** the port currents is simply the coherent **sum** of the voltage at that port due to **each** of the currents!

For example, the voltage at port 3 can be determined by:

$$V_3 = Z_{34}\, I_4 + Z_{33}\, I_3 + Z_{32}\, I_2 + Z_{31}\, I_1$$

More generally, the voltage at port *m* of an *N*-port device is:

$$V_m = \sum_{n=1}^{N} Z_{mn} I_n$$

This expression can be written in **matrix** form as:

$$\mathbf{V} = \mathcal{Z}\,\mathbf{I}$$

Where $\mathbf{I}$ is the **vector**:

$$\mathbf{I} = \left[I_1, I_2, I_3, \cdots, I_N \right]^T$$

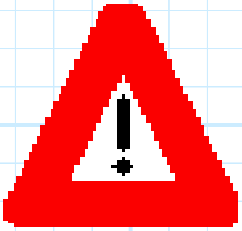
and **V** is the vector:

$$\mathbf{V} = \left[V_1, V_2, V_3, \ldots, V_N \right]^T$$

And the matrix $\mathcal{Z}$ is called the **impedance matrix**:

$$\mathcal{Z} = \begin{bmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & \ddots & \vdots \\ Z_{m1} & \cdots & Z_{mn} \end{bmatrix}$$

The impedance matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the impedance matrix describes a multi-port device the way that Z_L describes a single-port device (e.g., a load)!

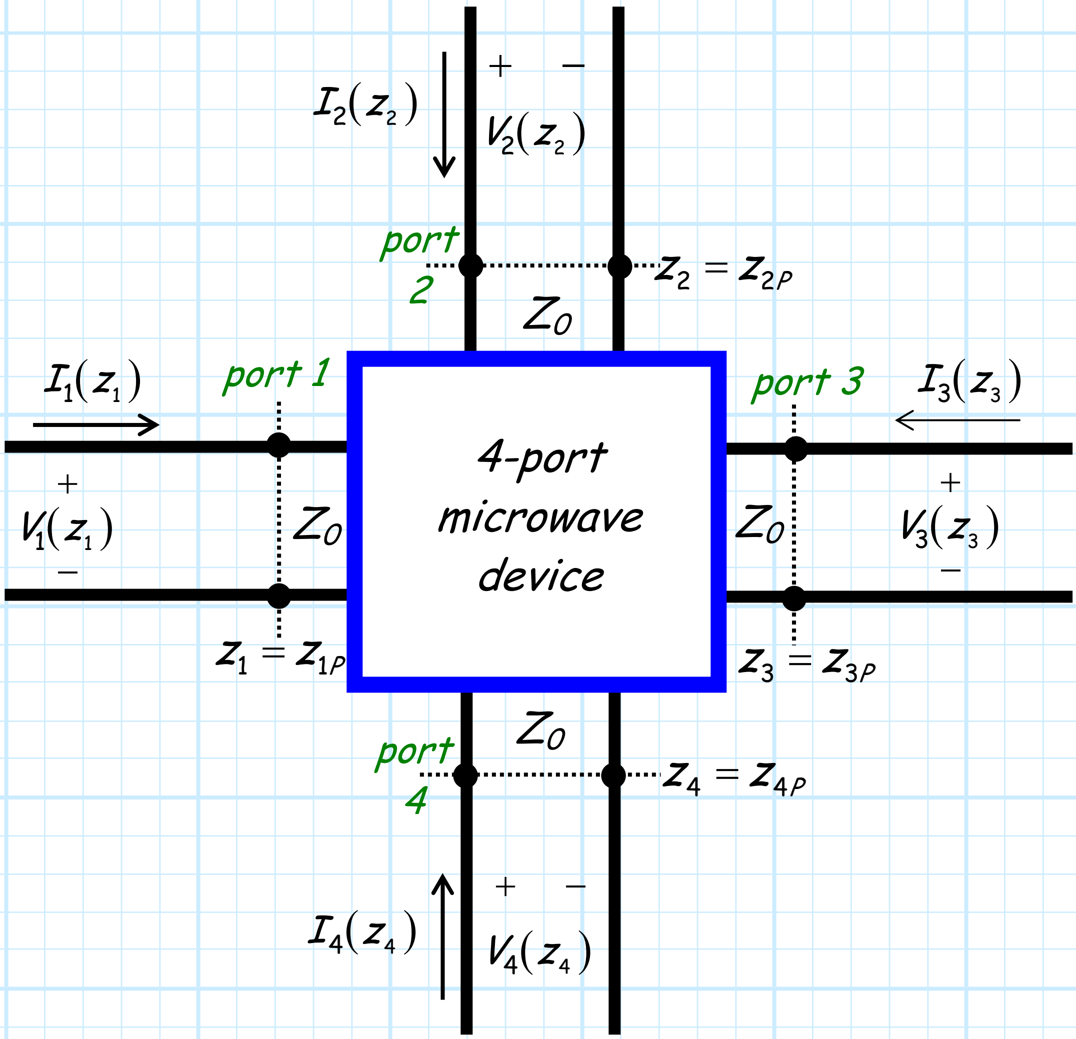


But **beware**! The values of the impedance matrix for a particular device or network, just like Z_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{Z}(\omega) = \begin{bmatrix} Z_{11}(\omega) & \dots & Z_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Z_{m1}(\omega) & \cdots & Z_{mn}(\omega) \end{bmatrix}$$

The Admittance Matrix

Consider again the **4-port** microwave device shown below:

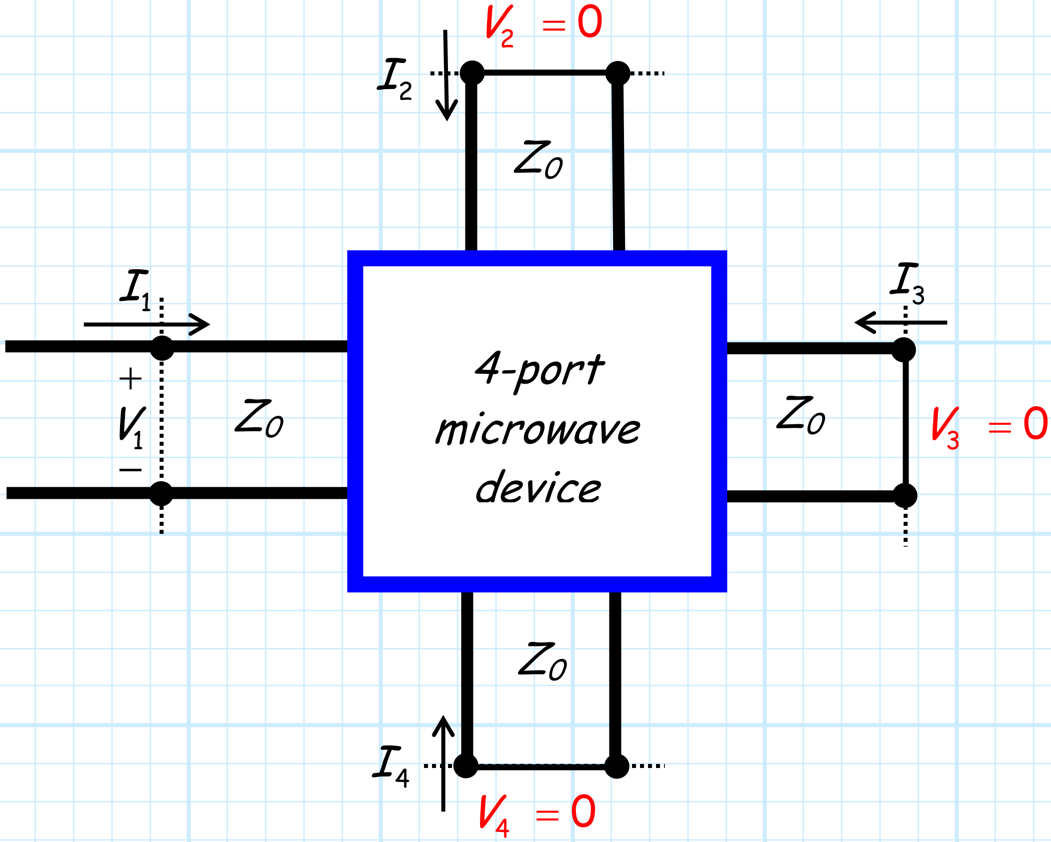


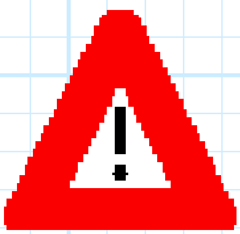
In addition to the Impedance Matrix, we can fully characterize this linear device using the **Admittance Matrix**.

The elements of the Admittance Matrix are the **trans-admittance** parameters Y_{mn}, defined as:

$$Y_{mn} = \frac{I_m}{V_n} \qquad \text{(given that } \; V_k = 0 \;\text{ for all } k \neq n\text{)}$$

Note here that the **voltage** at all but one port **must** be equal to **zero**. We can ensure that by simply placing a **short circuit** at these zero voltage ports!





Note that $Y_{mn} \neq 1/Z_{mn}$!

Now, we can thus **equivalently** state the definition of trans-admittance as:

$$Y_{mn} = \frac{V_m}{I_n} \qquad \text{(given that all ports } k \neq n \text{ are } \textbf{short - circuited}\text{)}$$

Just as with the trans-impedance values, we can use the trans-admittance values to evaluate general circuit problems, where **none** of the ports have zero voltage.

Since the device is **linear**, the current at any **one** port due to **all** the port currents is simply the coherent **sum** of the currents at that port due to **each** of the port voltages!

For example, the **current** at port 3 can be determined by:

$$I_3 = Y_{34} V_4 + Y_{33} V_3 + Y_{32} V_2 + Y_{31} V_1$$

More **generally**, the current at port m of an N-port device is:

$$I_m = \sum_{n=1}^{N} Y_{mn} V_n$$

This expression can be written in **matrix** form as:

$$\mathbf{I} = \mathcal{Y}\,\mathbf{V}$$

Where $\mathbf{I}$ is the **vector**:

$$\mathbf{I} = \left[I_1, I_2, I_3, \cdots, I_N\right]^T$$

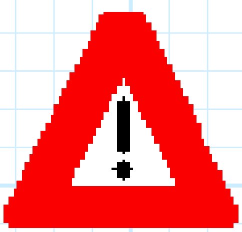
and $\mathbf{V}$ is the vector:

$$\mathbf{V} = \left[V_1, V_2, V_3, \ldots, V_N\right]^T$$

And the matrix $\mathcal{Y}$ is called the **admittance matrix**:

$$\mathcal{Y} = \begin{bmatrix} Y_{11} & \ldots & Y_{1n} \\ \vdots & \ddots & \vdots \\ Y_{m1} & \ldots & Y_{mn} \end{bmatrix}$$

The admittance matrix is a N by N matrix that **completely characterizes** a linear, N-port device. Effectively, the admittance matrix describes a multi-port device the way that Y_L describes a single-port device (e.g., a load)!



But **beware**! The values of the admittance matrix for a particular device or network, just like Y_L, are **frequency dependent**! Thus, it may be more instructive to **explicitly** write:

$$\mathcal{Y}(\omega) = \begin{bmatrix} Y_{11}(\omega) & \ldots & Y_{1n}(\omega) \\ \vdots & \ddots & \vdots \\ Y_{m1}(\omega) & \cdots & Y_{mn}(\omega) \end{bmatrix}$$

Q: *You said earlier that $Y_{mn} \neq 1/Z_{mn}$. Is there any **relationship** between the admittance and impedance matrix of a given device?*

A: I don't know! Let's see if **we** can figure it out.

Recall that we can determine the inverse of a matrix. Denoting the matrix inverse of the admittance matrix as $\mathcal{Y}^{-1}$, we find:

$$\mathbf{I} = \mathcal{Y}\,\mathbf{V}$$

$$\mathcal{Y}^{-1}\,\mathbf{I} = \mathcal{Y}^{-1}\left(\mathcal{Y}\,\mathbf{V}\right)$$

$$\mathcal{Y}^{-1}\,\mathbf{I} = \left(\mathcal{Y}^{-1}\,\mathcal{Y}\right)\mathbf{V}$$

$$\mathcal{Y}^{-1}\,\mathbf{I} = \mathbf{V}$$

Meaning that:

$$\mathbf{V} = \mathcal{Y}^{-1}\,\mathbf{I}$$

But, we likewise know that:

$$\mathbf{V} = \mathcal{Z}\,\mathbf{I}$$

By comparing the two previous expressions, we can conclude:

$$\mathcal{Z} = \mathcal{Y}^{-1} \qquad \text{and} \qquad \mathcal{Z}^{-1} = \mathcal{Y}$$

Reciprocal and Lossless Networks

We can **classify** multi-port devices or networks as either **lossless** or lossy; **reciprocal** or non-reciprocal. Let's look at each classification individually:

Lossless

A **lossless** network or device is simply one that **cannot** absorb power. This does **not** mean that the delivered power at **every port** is zero; rather, it means the total power flowing **into** the **device** must equal the total power **exiting** the **device**.

A lossless device exhibits an impedance matrix with an interesting **property**. Perhaps not surprisingly, we find for a lossless device that the **elements** of its impedance matrix will be **purely reactive**:

$$Re\{Z_{mn}\} = 0 \qquad \text{for a \textbf{lossless} device.}$$

If the device is lossy, then the elements of the impedance matrix must have **at least** one element with a real (i.e., resistive) component.

Moreover, we similarly find that if the elements of an **admittance** matrix are **all** purely imaginary (i.e., $Re\{Y_{mn}\}=0$), then the device is lossless.

Reciprocal

Generally speaking, most **passive, linear** microwave components will turn out to be **reciprocal**—regardless of whether the designer **intended** it to be or not!

Reciprocity is basically a "natural" effect of using simple linear materials such as **dielectrics** and **conductors**. It results from a characteristic in **electromagnetics** called "reciprocity"—a characteristic that is difficult to **prevent**!

But reciprocity is a tremendously important characteristic, as it greatly **simplifies** an impedance or admittance matrix!

Specifically, we find that a reciprocal device will result in a **symmetric** impedance and admittance **matrix**, meaning that:

$$Z_{mn} = Z_{nm} \qquad Y_{mn} = Y_{nm} \qquad \text{for reciprocal devices}$$

For **example**, we find for a reciprocal device that $Z_{23}=Z_{32}$, and $Y_{21}=Y_{12}$.

Let's illustrate these concepts with **four examples:**

$$\mathcal{Z} = \begin{bmatrix} j2 & 0.1 & j3 \\ -j & -1 & 1 \\ 4 & -2 & 0.5 \end{bmatrix}$$

Neither lossless **nor** reciprocal.

$$\mathcal{Z} = \begin{bmatrix} j2 & j0.1 & j3 \\ -j & -j1 & j1 \\ j4 & -j2 & j0.5 \end{bmatrix}$$

Lossless, but not reciprocal.

$$\mathcal{Z} = \begin{bmatrix} j2 & -j & 4 \\ -j & -1 & -j2 \\ 4 & -j2 & j0.5 \end{bmatrix}$$

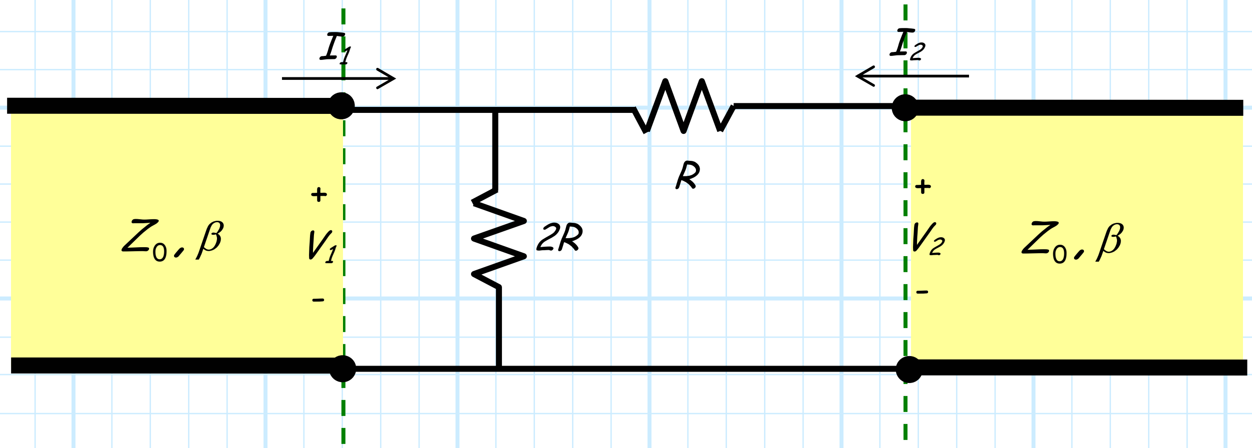
Reciprocal, but not lossless.

$$\mathcal{Z} = \begin{bmatrix} j2 & -j & j4 \\ -j & -j & -j2 \\ j4 & -j2 & j0.5 \end{bmatrix}$$

Both reciprocal **and** lossless.

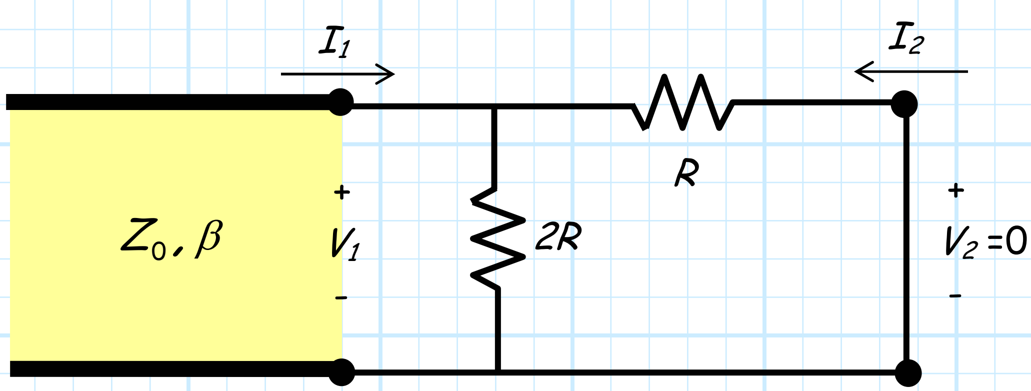
Example: Evaluating the Admittance Matrix

Consider the following **two-port** device:



Let's determine the **admittance matrix** of this device!

Step 1: Place a **short** at port 2.



Step 2: Determine **currents** I_1 and I_2 .

Note that **after** the short was placed at port 2, both resistors are in **parallel**, with a potential V_2 across each.

The current I_1 is thus simply the **sum** of the two currents through **each** resistor:

$$I_1 = \frac{V_1}{2R} + \frac{V_1}{R} = \frac{3V_1}{2R}$$

The current I_2 is simply the **opposite** of the current through R:

$$I_2 = -\frac{V_1}{R}$$

Step 3: Determine **trans-admittance** Y_{11} and Y_{21} .

$$Y_{11} = \frac{I_1}{V_1} = \frac{3}{2R}$$

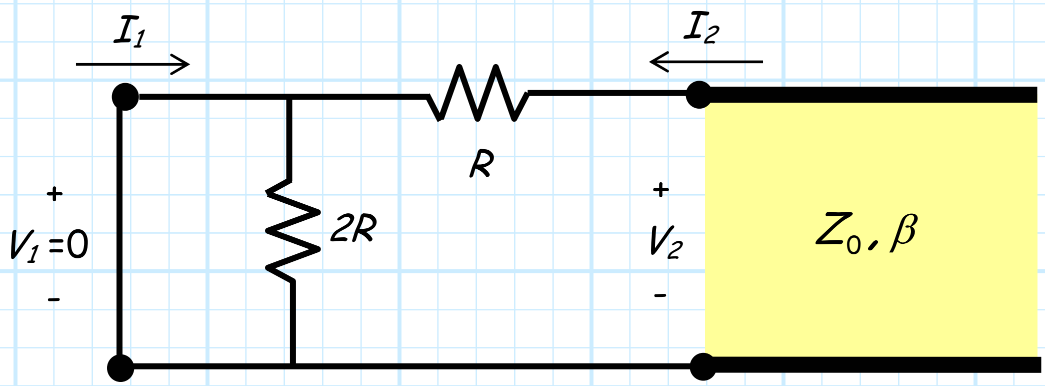
$$Y_{21} = \frac{I_2}{V_1} = -\frac{1}{R}$$

Note that Y_{21} is **real**—but **negative**!

This is **still** a valid physical result, **although** you will find that the **diagonal** terms of an impedance or admittance matrix (e.g., Y_{22}, Z_{11}, Y_{44}) will **always** have a real component that is **positive**.

To find the **other two** trans-admittance parameters, we must **move** the short and then **repeat** each of our previous steps!

Step 1: Place a **short** at port 1.



Step 2: Determine **currents** I_1 and I_2 .

Note that **after** a short was placed at port **1**, resistor *2R* has **zero** voltage across it—and thus **zero current** through it!

Likewise, from KVL we find that the **voltage** across resistor *R* is equal to V_2.

Finally, we see from KCL that $I_1 = I_2$.

The current I_2 thus:

$$I_2 = \frac{V_2}{R}$$

and thus:

$$I_1 = -\frac{V_2}{R}$$

Step 3: Determine **trans-admittance** Y_{12} and Y_{22} .

$$Y_{12} = \frac{I_1}{V_2} = -\frac{1}{R}$$

$$Y_{22} = \frac{I_2}{V_2} = \frac{1}{R}$$

The **admittance** matrix of this two-port device is therefore:

$$\mathcal{Y} = \frac{1}{R}\begin{bmatrix} 1.5 & -1 \\ -1 & 1 \end{bmatrix}$$

Note this device (as **you** may have suspected) is **lossy** and **reciprocal**.

Q: *What about the **impedance** matrix? How can we **determine** that?*

A: **One** way is simply determine the **inverse** of the admittance matrix above.

$$\begin{aligned} \mathcal{Z} &= \mathcal{Y}^{-1} \\ &= R\begin{bmatrix} 1.5 & -1 \\ -1 & 1 \end{bmatrix}^{-1} \\ &= R\begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \end{aligned}$$

A: **Another** way to determine the impedance matrix is simply to apply the **definition** of trans-impedance to **directly** determine the elements of the impedance matrix—**similar** to how we just determined the admittance matrix!

Specifically, follow these **steps**:

Step 1: Place an **open** at port 2 (or 1)

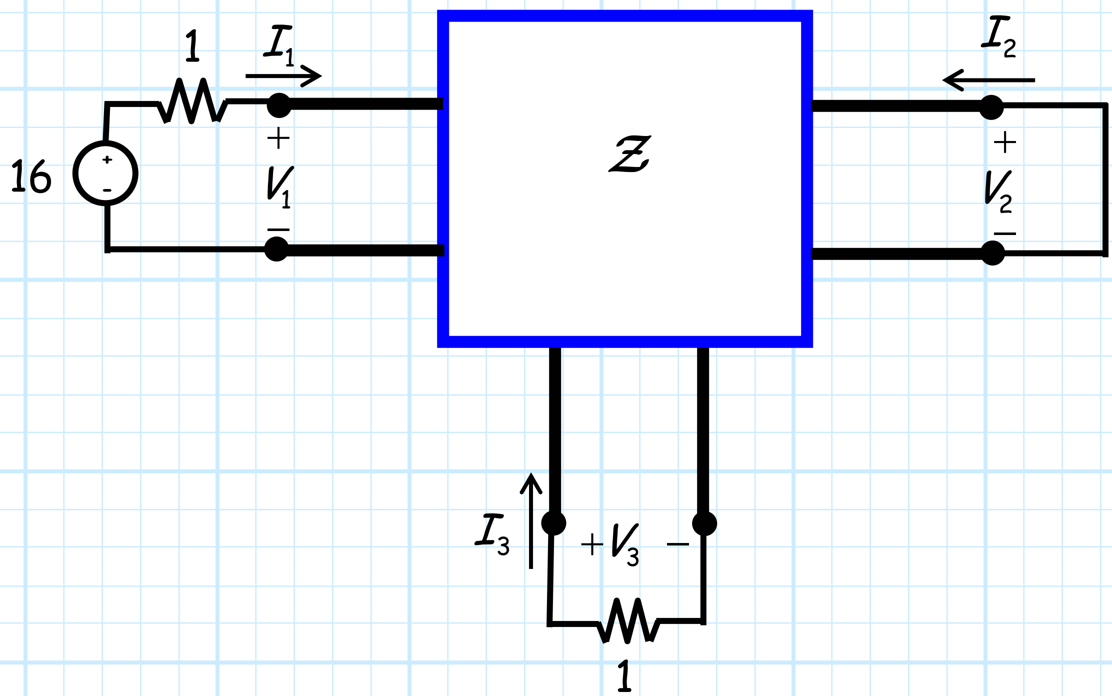
Step 2: Determine **voltages** V_1 and V_2 .

Step 3: Determine trans-**impedance** Z_{11} and Z_{21} (or Z_{12} and Z_{22}).

You try this procedure on the circuit of this example, and make sure **you** get the **same** result for $\mathcal{Z}$ as we determined on the previous page (from matrix inversion)—after all, **you** want to do **well** on my long, scary, evil **exam**!

Example: Using the Impedance Matrix

Consider the following circuit:



Where the 3-port **device** is characterized by the **impedance matrix**:

$$\mathcal{Z} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}$$

Let's now determine all port **voltages** V_1, V_2, V_3 and all **currents** I_1, I_2, I_3.

Q: *How can we do that—we **don't** know what the device is made of! What's **inside** that box?*

A: We **don't** need to know what's inside that box! We know its impedance matrix, and that **completely** characterizes the device (or, at least, characterizes it at **one** frequency).

Thus, we have enough information to solve this problem. From the impedance matrix we know:

$$V_1 = 2\,I_1 + I_2 + 2\,I_3$$

$$V_2 = I_1 + I_2 + 4\,I_3$$

$$V_3 = 2\,I_1 + 4\,I_2 + I_3$$

Q: *Wait! There are only **3** equations here, yet there are **6** unknowns!?*

A: True! The impedance matrix describes the device in the box, but it does **not** describe the devices **attached** to it. We require **more** equations to describe them.

1. The **source** at port 1 is described by the equation:

$$V_1 = 16.0 - (1)\, I_1$$

2. The **short** circuit on port 2 means that:

$$V_2 = 0$$

3. While the **load** on port 3 leads to:

$$V_3 = -(1)\, I_3 \quad \text{(note the \textbf{minus sign}!)}$$

Now we have **6** equations and **6** unknowns! Combining equations, we find:

$$V_1 = 16 - I_1 = 2\, I_1 + I_2 + 2\, I_3$$
$$\therefore \quad 16 = 3\, I_1 + I_2 + 2\, I_3$$

$$V_2 = 0 = I_1 + I_2 + 4\, I_3$$
$$\therefore 0 = I_1 + I_2 + 4\, I_3$$

$$V_3 = -I_3 = 2\, I_1 + 4\, I_2 + I_3$$
$$\therefore \quad 0 = 2\, I_1 + 4\, I_2 + 2 I_3$$

Solving, we find (I'll let **you** do the algebraic details!):

$$I_1 = 7.0 \qquad I_2 = -3.0 \qquad I_3 = -1.0$$

$$V_1 = 9.0 \qquad V_2 = 0.0 \qquad V_3 = 1.0$$