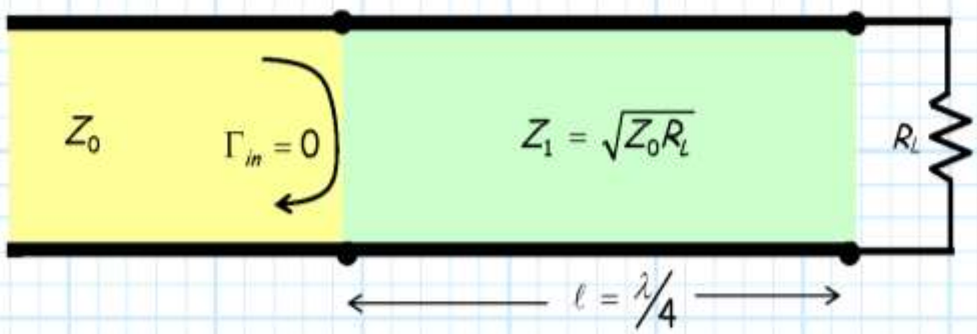
Multiple Reflection Viewpoint

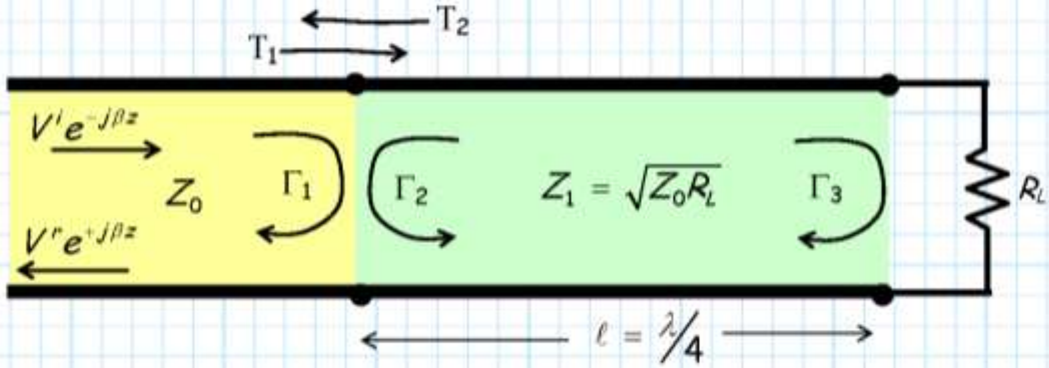
The **quarter-wave** transformer brings up an interesting question in μ-wave engineering.



Q: ***Why*** *is there* ***no*** *reflection at* $z = -\ell$ *? It appears that the line is* ***mismatched*** *at both* $z = 0$ *and* $z = -\ell$.

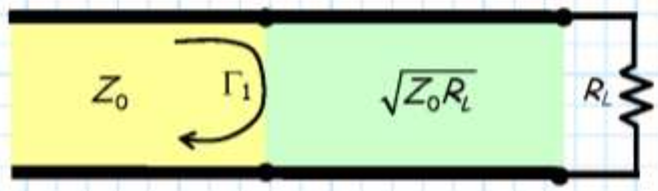
A: In fact there **are** reflections at these mismatched interfaces—an **infinite** number of them!

First, lets **define** a few terms:



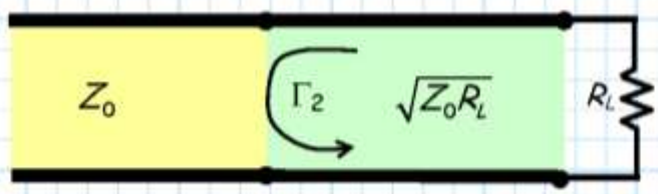
Γ_1 = **partial** reflection coefficient of a wave incident on the $z=-\ell$ interface from the Z_0 line:

$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}$$



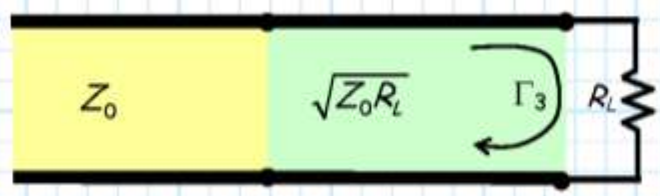
Γ_2 = **partial** reflection coefficient of a wave incident on the $z=-\ell$ interface from the Z_1 line:

$$\Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1$$



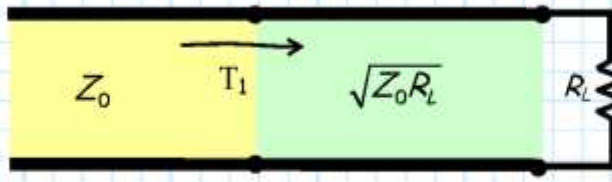
Γ_3 = **partial** reflection coefficient of a wave incident on the $z=-0$ interface from the Z_1 line:

$$\Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1}$$



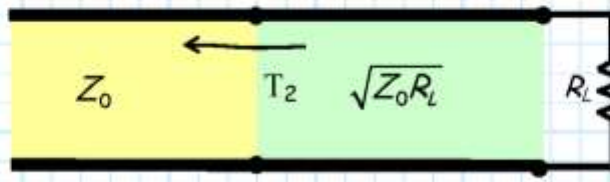
T_1 = **partial** transmission coefficient of a wave incident on the $z = -\ell$ interface from the Z_0 line:

$$T_1 = \frac{2Z_1}{Z_1 + Z_0}$$

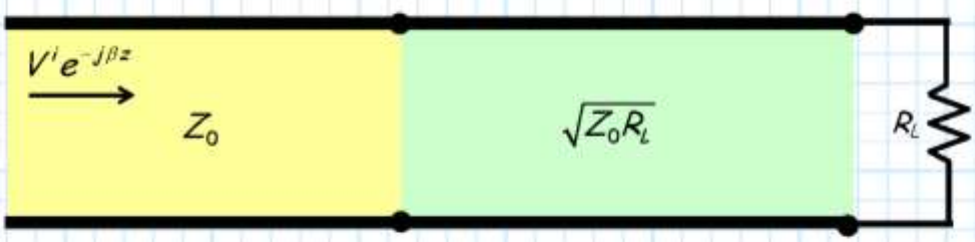


T_2 = **partial** transmission coefficient of a wave incident on the $z = -\ell$ interface from the Z_1 line:

$$T_2 = \frac{2Z_0}{Z_0 + Z_1}$$

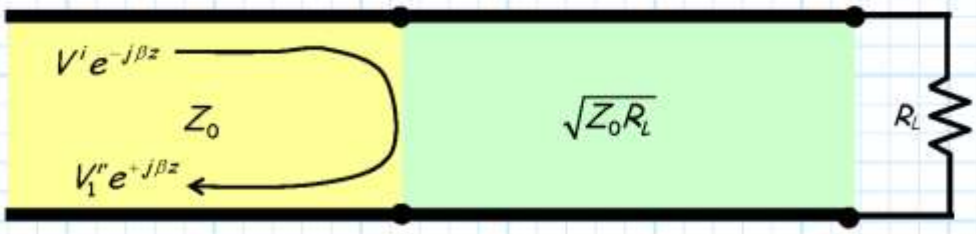


Now let's try to intemperate what **physically** happens when the **incident** voltage wave:



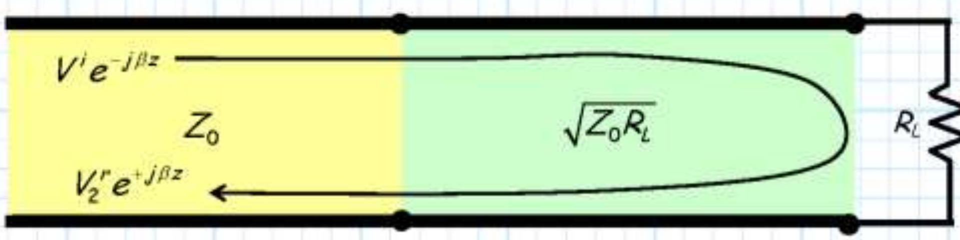
reaches the interface at $z = -\ell$.

1. At $z = -\ell$, the characteristic impedance of the transmission line changes from Z_0 to Z_1. This mismatch creates a **reflected** wave:



where $V_1^r = \Gamma_1 V^i$.

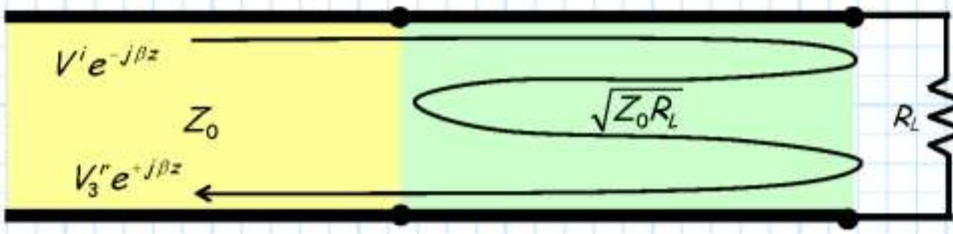
2. However, a **portion** of the incident wave is transmitted (T_1) across the interface at $z = -\ell$, this wave travels a distance of $\beta\ell = 90^\circ$ to the load at $z = 0$, where a portion of it is reflected (Γ_3). This wave travels back $\beta\ell = 90^\circ$ to the interface at $z = -\ell$, where a portion is again transmitted (T_2) across into the Z_0 transmission line—**another** reflected wave (V_2^r)!



where we have found that traveling $2\beta\ell = 180^\circ$ has produced a **minus** sign in our result:

$$\begin{aligned} V_2^r &= \mathrm{T}_2 \, e^{-j90^\circ} \, \Gamma_3 \, e^{-j90^\circ} \, \mathrm{T}_1 V^i \\ &= -\mathrm{T}_1 \, \mathrm{T}_2 \, \Gamma_3 V^i \end{aligned}$$

3. However, a **portion** of this **second** wave is also **reflected** (Γ_2) back into the Z_1 transmission line at $z=-\ell$, where it again travels to $\beta\ell = 90^\circ$ the load, is partially reflected (Γ_3), travels $\beta\ell = 90^\circ$ back to $z=-\ell$, and is partially transmitted into Z_0 (T_2)—our **third** reflected wave!



where:

$$
\begin{aligned}
V_3^r &= \mathrm{T}_2 e^{-j90^\circ}\,\Gamma_3 e^{-j90^\circ}\,\Gamma_2 e^{-j90^\circ}\,\Gamma_3 e^{-j90^\circ}\,\mathrm{T}_1 V^i \\
&= \mathrm{T}_1\,\mathrm{T}_2\,(\Gamma_3)^2\,\Gamma_2 V^i
\end{aligned}
$$

***n*.** We can see that this "bouncing" back and forth can go on **forever**, with each trip launching a **new** reflected wave into the Z_0 transmission line.

Note however, that the **power** associated with each successive reflected wave is **smaller** than the previous, and so eventually, the power associated with the reflected waves will **diminish** to insignificance!

Q: *But, why then is* $\Gamma = 0$ *?*

A: Each reflected wave V_n^r is a **coherent** wave. That is, they all oscillate at same frequency ω; the reflected waves differ only in terms of their **magnitude** and **phase**.

Therefore, to determine the **total** reflected wave, we must perform a **coherent summation** of each reflected wave—a operation easily performed since we have expressed our waves with **complex** notation:

$$V^r e^{+j\beta z} = \sum_{n=1}^{\infty} V_n^r e^{+j\beta z}$$

It can be shown that this infinite series **converges**, with the result:

$$V^r = \left(\frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - \mathrm{T}_1 \mathrm{T}_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3} \right) V^i$$

Thus, the **total** reflection coefficient is:

$$\Gamma = \frac{V^r}{V^i} = \frac{\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - \mathrm{T}_1 \mathrm{T}_2 \Gamma_3}{1 + \Gamma_2 \Gamma_3}$$

Using our definitions, it can likewise be shown that the **numerator** of the above expression is:

$$\Gamma_1 + \Gamma_1 \Gamma_2 \Gamma_3 - \mathrm{T}_1 \mathrm{T}_2 \Gamma_3 = \frac{2\left(Z_1^2 - Z_0 R_L\right)}{(Z_1 + Z_0)(R_L + Z_1)}$$

It is evident that the numerator (and therefore Γ) will be **zero** if:

$$Z_1^2 - Z_0 R_L \qquad \Rightarrow \qquad Z_1 = \sqrt{Z_0 R_L}$$

Just as we expected!

Physically, this results insures that all the reflected waves add coherently together to produce a **zero value**!

A simple example of this phenomenon is the addition of **two** waves with **equal** magnitude and **opposite** phase (i.e., their phase difference is 180°).

$$cos(\omega t) + cos(\omega t + 180^\circ) = cos(\omega t) - cos(\omega t) = 0$$

Note **all** of our transmission line analysis has been **steady-state** analysis. We assume our signals are **sinusoidal**, of the form $\exp(j\omega t)$. Note this signal exists for **all time** t—the signal is assumed to have been "on" **forever**, and assumed to continue on forever.