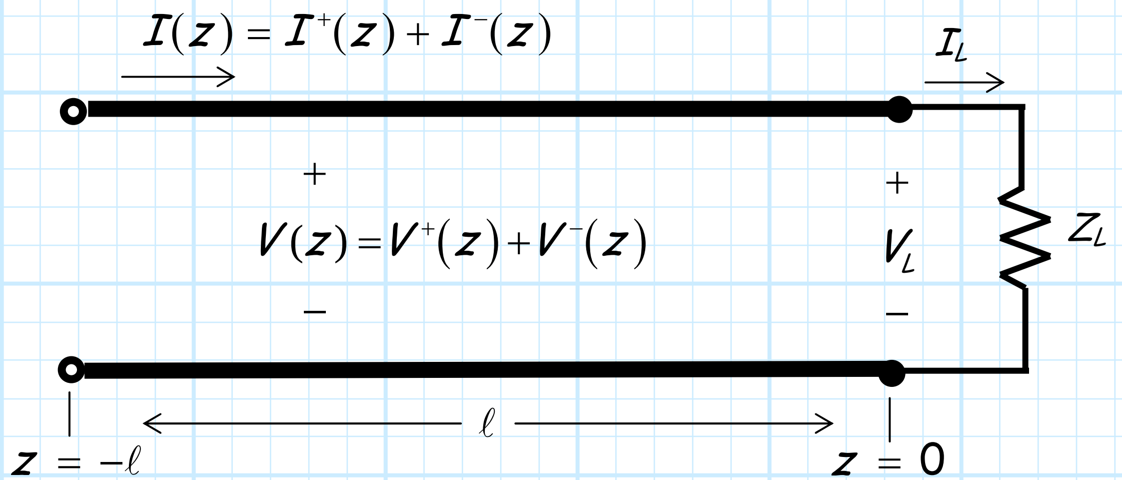
# Transmission Line Input Impedance

Consider a **lossless** line, length $\ell$, terminated with a **load** $Z_L$.




Let's determine the **input impedance** of this line!

**Q:** *Just what do you mean by **input** impedance?*

**A:** The input impedance is simply the line impedance seen at the **beginning** $(z = -\ell)$ of the transmission line, i.e.:

$$Z_{in} = Z(z = -\ell) = \frac{V(z = -\ell)}{I(z = -\ell)}$$

Note $Z_{in}$ equal to **neither** the load impedance $Z_L$ **nor** the characteristic impedance $Z_0$!

$$Z_{in} \neq Z_L \quad \text{and} \quad Z_{in} \neq Z_0$$

To determine exactly what $Z_{in}$ is, we first must determine the voltage and current at the **beginning** of the transmission line ($z = -\ell$).

$$V(z=-\ell) = V_0^+\left[e^{+j\beta\ell} + \Gamma_0\, e^{-j\beta\ell}\right]$$

$$I(z=-\ell) = \frac{V_0^+}{Z_0}\left[e^{+j\beta\ell} - \Gamma_0\, e^{-j\beta\ell}\right]$$

Therefore:

$$Z_{in} = \frac{V(z=-\ell)}{I(z=-\ell)} = Z_0\left(\frac{e^{+j\beta\ell} + \Gamma_0\, e^{-j\beta\ell}}{e^{+j\beta\ell} - \Gamma_0\, e^{-j\beta\ell}}\right)$$

We can explicitly write $Z_{in}$ in terms of load $Z_L$ using the previously determined relationship:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \Gamma_0$$

Combining these two expressions, we get:

$$\begin{aligned} Z_{in} &= Z_0\,\frac{(Z_L + Z_0)\,e^{+j\beta\ell} + (Z_L - Z_0)\,e^{-j\beta\ell}}{(Z_L + Z_0)\,e^{+j\beta\ell} - (Z_L - Z_0)\,e^{-j\beta\ell}} \\ &= Z_0\left(\frac{Z_L\left(e^{+j\beta\ell} + e^{-j\beta\ell}\right) + Z_0\left(e^{+j\beta\ell} - e^{-j\beta\ell}\right)}{Z_L\left(e^{+j\beta\ell} + e^{-j\beta\ell}\right) - Z_0\left(e^{+j\beta\ell} - e^{-j\beta\ell}\right)}\right) \end{aligned}$$

Now, recall **Euler's equations**:

$$e^{+j\beta\ell} = \cos\beta\ell + j\sin\beta\ell$$
$$e^{-j\beta\ell} = \cos\beta\ell - j\sin\beta\ell$$

Using Euler's relationships, we can likewise write the input impedance without the **complex** exponentials:

$$Z_{in} = Z_0 \left( \frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right)$$

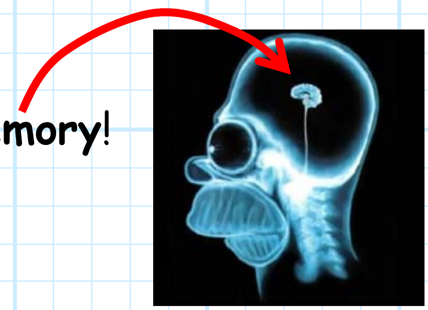
$$= Z_0 \left( \frac{Z_L + j\, Z_0 \tan\beta\ell}{Z_0 + j\, Z_L \tan\beta\ell} \right)$$

Note that depending on the values of $\beta$, $Z_0$ and $\ell$, the input impedance can be **radically** different from the load impedance $Z_L$!

## *Special Cases*

Now let's look at the $Z_{in}$ for some important **load** impedances and **line lengths**.

→ You should commit these results to **memory**!



**1.** $\ell = \lambda/2$

If the length of the transmission line is exactly **one-half** wavelength ($\ell = \lambda/2$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{2} = \pi$$

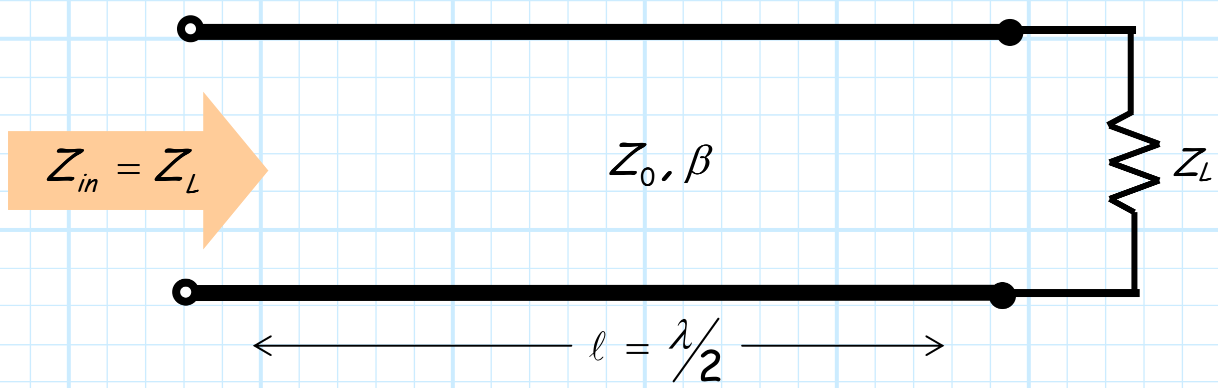
meaning that:

$$\cos\beta\ell = \cos\pi = -1 \qquad \text{and} \qquad \sin\beta\ell = \sin\pi = 0$$

and therefore:

$$\begin{aligned} Z_{in} &= Z_0 \left( \frac{Z_L \cos\beta\ell + j\, Z_0 \sin\beta\ell}{Z_0 \cos\beta\ell + j\, Z_L \sin\beta\ell} \right) \\ &= Z_0 \left( \frac{Z_L\,(-1) + j\, Z_L\,(0)}{Z_0\,(-1) + j\, Z_L\,(0)} \right) \\ &= Z_L \end{aligned}$$

In other words, if the transmission line is precisely **one-half wavelength** long, the **input** impedance is equal to the **load** impedance, **regardless** of $Z_0$ or $\beta$.




**2.** $\ell = \lambda/4$

If the length of the transmission line is exactly **one-quarter** wavelength ($\ell = \lambda/4$), we find that:

$$\beta\ell = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

meaning that:

$$\cos\beta\ell = \cos\ \pi/2 = 0 \qquad \text{and} \qquad \sin\beta\ell = \sin\ \pi/2 = 1$$

and therefore:

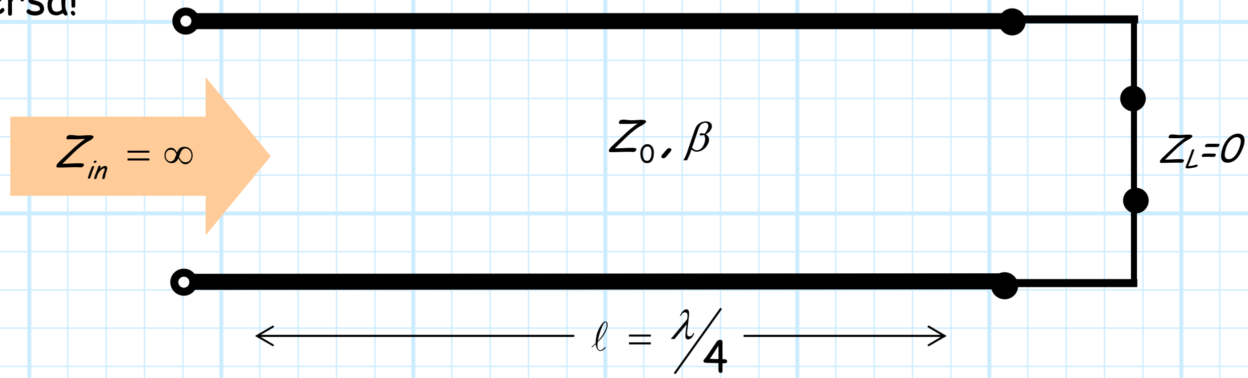
$$\begin{aligned} Z_{in} &= Z_0 \left( \frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right) \\ &= Z_0 \left( \frac{Z_L\,(0) + j\, Z_0\,(1)}{Z_0\,(0) + j\, Z_L\,(1)} \right) \\ &= \frac{(Z_0)^2}{Z_L} \end{aligned}$$

In other words, if the transmission line is precisely **one-quarter wavelength** long, the **input** impedance is **inversely** proportional to the **load** impedance.

Think about what this means! Say the load impedance is a **short** circuit, such that $Z_L = 0$. The **input impedance** at beginning of the $\lambda/4$ transmission line is therefore:

$$Z_{in} = \frac{(Z_0)^2}{Z_L} = \frac{(Z_0)^2}{0} = \infty$$

$Z_{in} = \infty$ ! This is an **open** circuit! The quarter-wave transmission line **transforms** a short-circuit into an open-circuit—and vice versa!

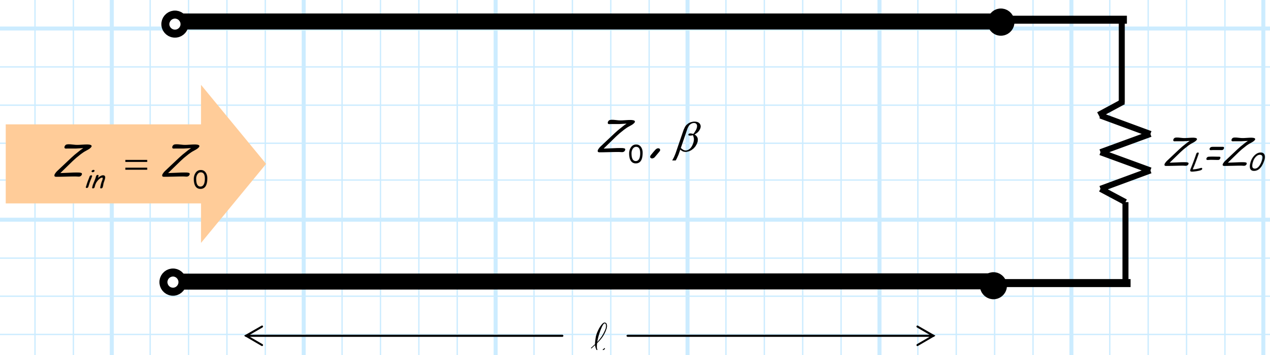


**3.** $Z_L = Z_0$

If the load is **numerically equal** to the characteristic impedance of the transmission line (a real value), we find that the input impedance becomes:

$$\begin{aligned} Z_{in} &= Z_0 \left( \frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right) \\ &= Z_0 \left( \frac{Z_0 \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_0 \sin \beta\ell} \right) \\ &= Z_0 \end{aligned}$$

In other words, if the **load** impedance is equal to the transmission line **characteristic** impedance, the **input** impedance will be likewise be equal to $Z_0$ regardless of the transmission line length $\ell$.

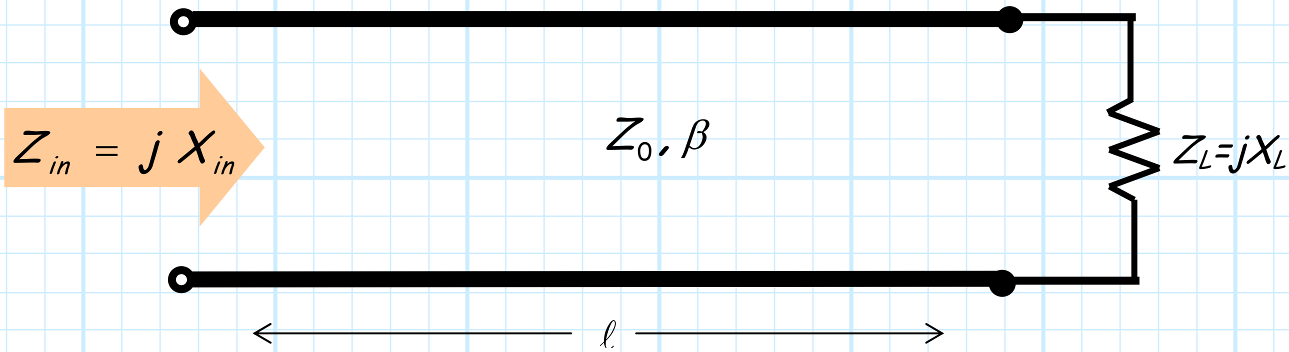



**4.** $Z_L = j\,X_L$

If the load is **purely reactive** (i.e., the resistive component is zero), the input impedance is:

$$
\begin{aligned}
Z_{in} &= Z_0 \left( \frac{Z_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j\, Z_L \sin \beta\ell} \right) \\
&= Z_0 \left( \frac{j\, X_L \cos \beta\ell + j\, Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell + j^2\, X_L \sin \beta\ell} \right) \\
&= j\, Z_0 \left( \frac{X_L \cos \beta\ell + Z_0 \sin \beta\ell}{Z_0 \cos \beta\ell - X_L \sin \beta\ell} \right)
\end{aligned}
$$

In other words, if the load is purely reactive, then the input impedance will **likewise** be purely reactive, **regardless** of the line length $\ell$.




Note that the **opposite** is **not** true: even if the load is **purely resistive** ($Z_L = R$), the input impedance will be **complex** (both resistive and reactive components).

**Q:** *Why is this?*

**A:**